

January 14, 1875.

JOSEPH DALTON HOOKER, C.B., President, in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

- I. "On a Class of Identical Relations in the Theory of Elliptic Functions." By J. W. L. GLAISHER, M.A., Fellow of Trinity College, Cambridge. Communicated by JAMES GLAISHER, F.R.S. Received November 23, 1874.

(Abstract.)

The object of the memoir is to notice certain forms into which the primary elliptic functions admit of being thrown, and to discuss the identical relations to which they give rise. These latter, it is shown, can be obtained directly by the aid of Fourier's theorem, or in a less straightforward manner by ordinary algebra.

Thus, *ex. gr.*, consider the cosine-amplitude: it is shown that we have the formula

$$\cos am 2Kx = \frac{\pi}{kK'} \left\{ \frac{1}{r^x + r^{-x}} - \frac{1}{r^{x-1} + r^{-(x-1)}} - \frac{1}{r^{x+1} + r^{-(x+1)}} \right. \\ \left. + \frac{1}{r^{x-2} + r^{-(x-2)}} + \frac{1}{r^{x+2} + r^{-(x+2)}} - \&c. \right\}$$

(r being $= e^{-\frac{\pi K}{K'}}$), which is noticeable as being of the form

$$\phi x - \phi(x-1) - \phi(x+1) + \&c.,$$

and therefore an analogue of

$$\pi \operatorname{cosec} \pi x = \frac{1}{x} - \frac{1}{x-1} - \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} - \&c.$$

This form of the cosine-amplitude gives rise to the identical equation

$$\operatorname{sech} x - \operatorname{sech}(x-\mu) - \operatorname{sech}(x+\mu) + \operatorname{sech}(x-2\mu) + \operatorname{sech}(x+2\mu) - \&c. \\ = \frac{2\pi}{\mu} \left\{ \operatorname{sech} \frac{\pi^2}{2\mu} \cos \frac{\pi x}{\mu} + \operatorname{sech} \frac{3\pi^2}{2\mu} \cos \frac{3\pi x}{\mu} + \&c. \right\} \dots (1)$$

(sech being the hyperbolic secant). The result (1) is deducible at once from the integral

$$\int_0^\infty \operatorname{sech} x \cos nx \, dx = \frac{\pi}{2} \operatorname{sech} \frac{n\pi}{2};$$

and it is remarked that all evaluable integrals of the forms

$$\int_0^\infty \phi x \cos nx \, dx, \quad \int_0^\infty \psi x \sin nx \, dx$$

(ϕ even and ψ uneven) give rise to identities of the same class as (1), and which, it appears, are all elliptic-function relations. It is also pointed out how (1) and the other similar identities discussed in the memoir admit of being simply established by ordinary algebra and trigonometry in two different ways.

Other identities and formulæ are noticed and compared; *ex. gr.*, for the cosine-amplitude, writing

$$x = \frac{\pi u}{2K}, \quad z = \frac{\pi u}{2K'}, \quad q = e^{-\frac{\pi K'}{K}} = e^{-\mu}, \quad r = e^{-\frac{\pi K}{K'}} = e^{-\nu},$$

we have

$$\begin{aligned} \cos am u &= \frac{2\pi}{kK} \left\{ \frac{q^{\frac{1}{2}}}{1+q} \cos x + \frac{q^{\frac{3}{2}}}{1+q^3} \cos 3x + \frac{q^{\frac{5}{2}}}{1+q^5} \cos 5x + \&c. \right\} \\ &= \frac{2\pi}{kK} \cos x \left\{ \frac{q^{\frac{1}{2}}(1-q)}{1-2q \cos 2x + q^2} - \frac{q^{\frac{3}{2}}(1-q^3)}{1-2q^3 \cos 2x + q^6} + \&c. \right\} \\ &= \frac{\pi}{2kK'} \{ \operatorname{sech} z - \operatorname{sech}(z-\nu) - \operatorname{sech}(z+\nu) + \&c. \} \\ &= \frac{\pi}{2kK'} \left\{ \operatorname{sech} z - 4 \cosh x \left(\frac{\cosh \nu}{\cosh 2z + \cosh 2\nu} \right. \right. \\ &\quad \left. \left. - \frac{\cosh 2\nu}{\cosh 4z + \cosh 4\nu} + \&c. \right) \right\} \\ &= \frac{\pi}{kK'} \left\{ \frac{\sinh(\frac{1}{2}\nu - z)}{\cosh \frac{1}{2}\nu} - \frac{\sinh 3(\frac{1}{2}\nu - z)}{\cosh \frac{3}{2}\nu} + \frac{\sinh 5(\frac{1}{2}\nu - z)}{\cosh \frac{5}{2}\nu} - \&c. \right\} \\ &= \frac{\pi}{2kK'} \left\{ \operatorname{sech} z - \frac{4r}{1+r} \cosh z + \frac{4r^3}{1+r^3} \cosh 3z - \&c. \right\}; \end{aligned}$$

so that if x, z, μ, ν be any four quantities subject to the relations

$$\mu\nu = \pi^2, \quad z = \frac{\pi x}{\mu} \left(\text{so that } x = \frac{\pi z}{\nu} \right),$$

the identities are

$$\begin{aligned} &\operatorname{sech} x - \operatorname{sech}(x-\mu) - \operatorname{sech}(x+\mu) + \operatorname{sech}(x-2\mu) \\ &\quad + \operatorname{sech}(x+2\mu) - \&c. \\ &= \operatorname{sech} x - 4 \cosh x \left(\frac{\cosh \mu}{\cosh 2x + \cosh 2\mu} - \frac{\cosh 2\mu}{\cosh 4x + \cosh 4\mu} + \&c. \right) \\ &= 2 \left\{ \frac{\sinh(\frac{1}{2}\mu - x)}{\cosh \frac{1}{2}\mu} - \frac{\sinh 3(\frac{1}{2}\mu - x)}{\cosh \frac{3}{2}\mu} \right. \\ &\quad \left. + \frac{\sinh 5(\frac{1}{2}\mu - x)}{\cosh \frac{5}{2}\mu} - \&c. \right\} \dots \dots \dots (2) \\ &= \operatorname{sech} x - \frac{4 \cosh x}{e^\mu + 1} + \frac{4 \cosh 3x}{e^{3\mu} + 1} - \&c. \\ &= \frac{2\pi}{\mu} \left\{ \frac{\cosh z}{\cosh \frac{1}{2}\nu} + \frac{\cosh 3z}{\cosh \frac{3}{2}\nu} + \frac{\cosh 5z}{\cosh \frac{5}{2}\nu} + \&c. \right\} \\ &= \frac{2\pi}{\mu} \cos z \left\{ \frac{\sinh \frac{1}{2}\nu}{\sin^2 x + \sinh^2 \frac{1}{2}\nu} - \frac{\sinh \frac{3}{2}\nu}{\sin^2 x + \sinh^2 \frac{3}{2}\nu} + \&c. \right\} \end{aligned}$$

(see Schellbach, 'Die Lehre von den elliptischen Integralen,' 1864, p. 113). In (2), x must be less than μ .

The two formulæ obtained in the memoir,

$$\begin{aligned} & \tanh x + \tanh (x - \mu) + \tanh (x + \mu) + \&c. \\ &= \frac{2x}{\mu} + \frac{2\pi}{\mu} \left\{ \frac{\sin 2z}{\sinh \nu} + \frac{\sin 4z}{\sinh 2\nu} + \&c. \right\}, \end{aligned}$$

$$\coth x + \coth (x - \mu) + \coth (x + \mu) + \&c.$$

$$= \frac{2x}{\mu} + \frac{\pi}{\mu} \left\{ \cot z + \frac{4 \sin 2z}{e^{2\nu} - 1} + \frac{4 \sin 4z}{e^{4\nu} - 1} + \&c., \right.$$

are considered in some detail, and an algebraical proof is added of Abel's identity ('Œuvres,' t. i. p. 307),

$$\frac{1}{\sqrt[24]{q}}(1+q)(1+q^3)(1+q^5)\dots = \frac{1}{\sqrt[24]{r}}(1+r)(1+r^3)(1+r^5)\dots$$

II. "On some remarkable Changes produced in Iron and Steel by the Action of Hydrogen and Acids." By WILLIAM H. JOHNSON, B.Sc. Communicated by Prof. Sir WILLIAM THOMSON, LL.D., F.R.S. Received December 7, 1874.

Some three years ago my attention was called to a remarkable change in some of the physical properties of iron caused by its temporary immersion in hydrochloric and sulphuric acids. This change is at once made evident to any one by the extraordinary decrease in toughness and breaking-strain of the iron so treated, and is all the more remarkable as it is not permanent, but only temporary in character, for with lapse of time the metal slowly regains its original toughness and strength. With a view of ascertaining the cause and degree of this change, I have from time to time made a number of experiments, some of which were carried out on a large scale in an iron-works where quantities of sulphuric and hydrochloric acids are used to remove the coating of oxide from iron wire, preparatory to drawing it. Many of these experiments have been already described in a somewhat desultory form in the 'Proceedings of the Literary and Philosophical Society of Manchester' for Jan. 7th, March 4th, Dec. 30th, 1873, Jan. 13th, March 10th and 24th, 1874.

As mentioned before, I first noticed that iron wire became more brittle after a few minutes' immersion (half a minute will sometimes suffice) in strong hydrochloric or dilute sulphuric acid—a piece breaking after being bent once on itself, while before immersion it would bear bending on itself and back again two or three times before breaking. But perhaps the most remarkable phenomenon was, that if, while still hot from the effort of breaking, the fractured part was wetted, it appeared to froth,